(b) The required line passes through the points with position vectors $-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}$ and $\boldsymbol{i} + 6\,\boldsymbol{j} - \boldsymbol{k}$.

Hence, the direction vector of the line is of the form

$(-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}) - (\boldsymbol{i} + 6\,\boldsymbol{j} - \boldsymbol{k}) = -3\,\boldsymbol{i} - 3\,\boldsymbol{j} + 7\,\boldsymbol{k}$

Therefore a possible vector equation of the required line is

$\boldsymbol{r} = (-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}) + \lambda\,(-3\,\boldsymbol{i} - 3\,\boldsymbol{j} + 7\,\boldsymbol{k}) = (-2 - 3\lambda)\,\boldsymbol{i} + (3 - 3\lambda)\,\boldsymbol{j} + (6 + 7\lambda)\,\boldsymbol{k}$

Example 2.2

Determine if the point with position vector $< -1, -6, 5 >$ lies on the line $\boldsymbol{r} = < -1, 2\, -3 > + \lambda < 0, -5, 5 >$.

Solution:

Since $\boldsymbol{r}$ *represents the position vector of any point on the line, then* $\boldsymbol{r} = < -1, -6, 5 >$ *must satisfy* $\boldsymbol{r} = < -1, 2\, -3 > + \lambda < 0, -5, 6 >$.

Consider $< -1, -6, 5 > = < -1, 2\, -3 > + \lambda < 0, -5, 5 >$.

Comparing $\boldsymbol{i}$-components: $-1 = -1$

Comparing $\boldsymbol{j}$-components: $2 - 5\lambda = -6 \;\Rightarrow\; \lambda = 1.6$

Comparing $\boldsymbol{k}$-components: $-3 + 5\lambda = 5 \;\Rightarrow\; \lambda = 1.6$

Therefore, $\boldsymbol{r} = < -1, -6, 5 >$ satisfies the equation $\boldsymbol{r} = < -1, 2\, -3 > + \lambda < 0, -5, 5 >$.

That is, the point with position vector $< -1, -6, 5 >$ lies on the given line.

Example 2.3

Use a vector method to find the position vector of the point of intersection between the lines $\boldsymbol{r} = < -1, 1, 3 > + \lambda < 1, 2, 1 >$ and $\boldsymbol{r} = < 2, 1, 8 > + \lambda < 1, -1, 2 >$.

Solution:

Rewrite equations as $\boldsymbol{r} = < -1, 1, 3 > + \lambda < 1, 2, 1 >$

$= < -1 + \lambda, 1 + 2\lambda, 3 + \lambda >$

and $\boldsymbol{r} = < 2, 1, 8 > + \mu < 1, -1, 2 >$

$= < 2 + \mu, 1 - \mu, 8 + 2\mu >$.

At the point of intersection,

$< -1 + \lambda, 1 + 2\lambda, 3 + \lambda > = < 2 + \mu, 1 - \mu, 8 + 2\mu >$.

Comparing components:

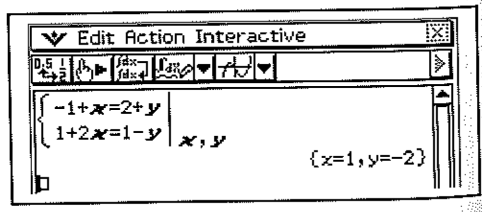
$-1 + \lambda = 2 + \mu$ (I)

$1 + 2\lambda = 1 - \mu$ (II)

$3 + \lambda = 8 + 2\mu$ (III)

The two lines are traced as the value of λ changes. However, they do not have to "trace" at the same pace. Hence, start by changing the "λ" in the second equation into "μ". This makes sure that the two lines trace independently.

Solve I and II simultaneously: $\lambda = 1, \mu = -2$.



Substitute $\lambda = 1, \mu = -2$ into (III),

a true statement is obtained.

Hence, the two lines meet at $< 0, 3, 4 >$.

Example 2.4

Use a vector method to show that the lines with equations $\boldsymbol{r} = < 1, -1, 2 > + \lambda < -2, 1, 1 >$ and $\boldsymbol{r} = < 1, 1, -1 > + \lambda < 3, -1, 1 >$ are non-intersecting.

Solution:

Rewrite equations as $\boldsymbol{r} = < 1 - 2\lambda, -1 + \lambda, 2 + \lambda >$

and $\boldsymbol{r} = < 1 + 3\mu, 1 - \mu, -1 + \mu >$.

At the point of intersection,

$< 1 - 2\lambda, -1 + \lambda, 2 + \lambda > = < 1 + 3\mu, 1 - \mu, -1 + \mu >$.

Comparing components:

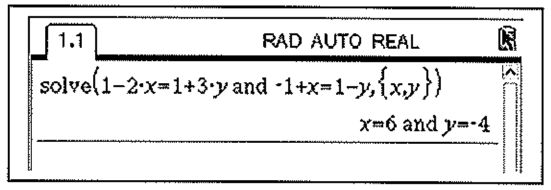
$1 - 2\lambda = 1 + 3\mu$ (I)

$-1 + \lambda = 1 - \mu$ (II)

$2 + \lambda = -1 + \mu$ (III)

The two lines are traced as the value of λ changes. However, they do not have to "trace" at the same pace. Hence, start by changing the "λ" in the second equation into "μ". This makes sure that the two lines trace independently.

Solve I and II simultaneously: $\lambda = 6, \mu = -4$.



Substitute $\lambda = 6, \mu = -4$ into (III),

a false statement is obtained.

Hence, the two lines do not intersect.

2.1.1 Parametric Equation of a Line

- The vector equation of the line L passing through the point with position vector $\boldsymbol{a}$ and parallel to vector $\boldsymbol{d}$ is given by $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$.

- Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

  Since, $\boldsymbol{r}$ represents the position vector of any point along this line, let $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

- Hence, the equation of the line can be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

- Comparing the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ components:

$$x = a_1 + \lambda d_1$$
$$y = a_2 + \lambda d_2$$
$$z = a_3 + \lambda d_3$$

This set of three equations is referred to as the parametric equation of the line L.

Example 2.4

Use a vector method to show that the lines with equations $\boldsymbol{r} = \langle 1, -1, 2 \rangle + \lambda \langle -2, 1, 1 \rangle$ and $\boldsymbol{r} = \langle 1, 1, -1 \rangle + \lambda \langle 3, -1, 1 \rangle$ are non-intersecting.

Solution:

Rewrite equations as $\boldsymbol{r} = \langle 1 - 2\lambda, -1 + \lambda, 2 + \lambda \rangle$

and $\boldsymbol{r} = \langle 1 + 3\mu, 1 - \mu, -1 + \mu \rangle$.

At the point of intersection,

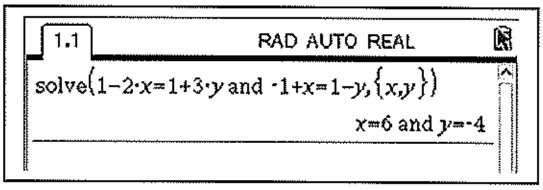
$$\langle 1 - 2\lambda, -1 + \lambda, 2 + \lambda \rangle = \langle 1 + 3\mu, 1 - \mu, -1 + \mu \rangle .$$

Comparing components:

$$1 - 2\lambda = 1 + 3\mu \qquad \text{(I)}$$
$$-1 + \lambda = 1 - \mu \qquad \text{(II)}$$
$$2 + \lambda = -1 + \mu \qquad \text{(III)}$$

The two lines are traced as the value of λ changes. However, they do not have to "trace" at the same pace. Hence, start by changing the "λ" in the second equation into "μ". This makes sure that the two lines trace independently.

Solve I and II simultaneously: $\lambda = 6$, $\mu = -4$.



Substitute $\lambda = 6$, $\mu = -4$ into (III), a false statement is obtained.

Hence, the two lines do not intersect.

2.1.1 Parametric Equation of a Line

- The vector equation of the line L passing through the point with position vector $\boldsymbol{a}$ and parallel to vector $\boldsymbol{d}$ is given by $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$.

- Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

  Since, $\boldsymbol{r}$ represents the position vector of any point along this line, let $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

- Hence, the equation of the line can be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

- Comparing the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ components:

$$x = a_1 + \lambda d_1$$
$$y = a_2 + \lambda d_2$$
$$z = a_3 + \lambda d_3$$

  This set of three equations is referred to as the parametric equation of the line L.

Example 2.4

Use a vector method to show that the lines with equations $\boldsymbol{r} = \langle 1, -1, 2 \rangle + \lambda \langle -2, 1, 1 \rangle$ and $\boldsymbol{r} = \langle 1, 1, -1 \rangle + \lambda \langle 3, -1, 1 \rangle$ are non-intersecting.

Solution:

Rewrite equations as $\boldsymbol{r} = \langle 1 - 2\lambda, -1 + \lambda, 2 + \lambda \rangle$
and $\boldsymbol{r} = \langle 1 + 3\mu, 1 - \mu, -1 + \mu \rangle$.

At the point of intersection,

$$\langle 1 - 2\lambda, -1 + \lambda, 2 + \lambda \rangle = \langle 1 + 3\mu, 1 - \mu, -1 + \mu \rangle.$$

Comparing components:

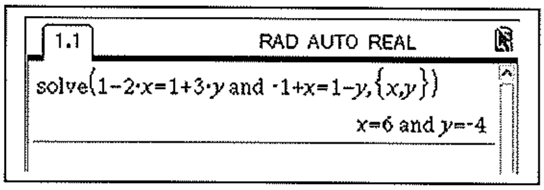
$$1 - 2\lambda = 1 + 3\mu \quad \text{(I)}$$
$$-1 + \lambda = 1 - \mu \quad \text{(II)}$$
$$2 + \lambda = -1 + \mu \quad \text{(III)}$$

The two lines are traced as the value of λ changes. However, they do not have to "trace" at the same pace. Hence, start by changing the "λ" in the second equation into "μ". This makes sure that the two lines trace independently.

Solve I and II simultaneously: $\lambda = 6, \mu = -4$.

Substitute $\lambda = 6, \mu = -4$ into (III), a false statement is obtained.

Hence, the two lines do not intersect.



2.1.1 Parametric Equation of a Line

- The vector equation of the line L passing through the point with position vector $\boldsymbol{a}$ and parallel to vector $\boldsymbol{d}$ is given by $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$.

- Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

  Since, $\boldsymbol{r}$ represents the position vector of any point along this line, let $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

- Hence, the equation of the line can be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

- Comparing the $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ components:

$$x = a_1 + \lambda d_1$$
$$y = a_2 + \lambda d_2$$
$$z = a_3 + \lambda d_3$$

This set of three equations is referred to as the parametric equation of the line L.

(b) The required line passes through the points with position vectors $-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}$ and $\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$.
Hence, the direction vector of the line is of the form
$(-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}) - (\boldsymbol{i} + 6\,\boldsymbol{j} - \boldsymbol{k}) = -3\,\boldsymbol{i} - 3\,\boldsymbol{j} + 7\,\boldsymbol{k}$
Therefore a possible vector equation of the required line is
$\boldsymbol{r} = (-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}) + \lambda\,(-3\,\boldsymbol{i} - 3\,\boldsymbol{j} + 7\,\boldsymbol{k}) = (-2 - 3\lambda)\,\boldsymbol{i} + (3 - 3\lambda)\,\boldsymbol{j} + (6 + 7\lambda)\,\boldsymbol{k}$

Example 2.2

Determine if the point with position vector $< -1, -6, 5 >$ lies on the line
$\boldsymbol{r} = < -1, 2\ -3 > + \lambda < 0, -5, 5 >$.

Solution:

Since $\boldsymbol{r}$ *represents the position vector of any point on the line, then* $\boldsymbol{r} = < -1, -6, 5 >$ *must satisfy*
$\boldsymbol{r} = < -1, 2\ -3 > + \lambda < 0, -5, 6 >$.

Consider $< -1, -6, 5 > = < -1, 2\ -3 > + \lambda < 0, -5, 5 >$.
Comparing $\boldsymbol{i}$-components: $-1 = -1$
Comparing $\boldsymbol{j}$-components: $2 - 5\lambda = -6 \Rightarrow \lambda = 1.6$
Comparing $\boldsymbol{k}$-components: $-3 + 5\lambda = 5 \Rightarrow \lambda = 1.6$

Therefore, $\boldsymbol{r} = < -1, -6, 5 >$ satisfies the equation $\boldsymbol{r} = < -1, 2\ -3 > + \lambda < 0, -5, 5 >$.
That is, the point with position vector $< -1, -6, 5 >$ lies on the given line.

Example 2.3

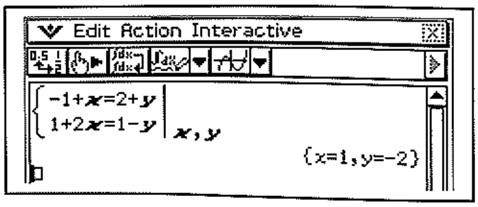
Use a vector method to find the position vector of the point of intersection between the lines $\boldsymbol{r} = < -1, 1, 3 > + \lambda < 1, 2, 1 >$ and $\boldsymbol{r} = < 2, 1, 8 > + \lambda < 1, -1, 2 >$.

Solution:

The two lines are traced as the value of λ changes. However, they do not have to "trace" at the same pace. Hence, start by changing the "λ" in the second equation into "μ". This makes sure that the two lines trace independently.

Rewrite equations as $\boldsymbol{r} = < -1, 1, 3 > + \lambda < 1, 2, 1 >$
$= < -1 + \lambda, 1 + 2\lambda, 3 + \lambda >$
and $\boldsymbol{r} = < 2, 1, 8 > + \mu < 1, -1, 2 >$
$= < 2 + \mu, 1 - \mu, 8 + 2\mu >$.
At the point of intersection,
$< -1 + \lambda, 1 + 2\lambda, 3 + \lambda > = < 2 + \mu, 1 - \mu, 8 + 2\mu >$.
Comparing components: $-1 + \lambda = 2 + \mu$ (I)
$1 + 2\lambda = 1 - \mu$ (II)
$3 + \lambda = 8 + 2\mu$ (III)

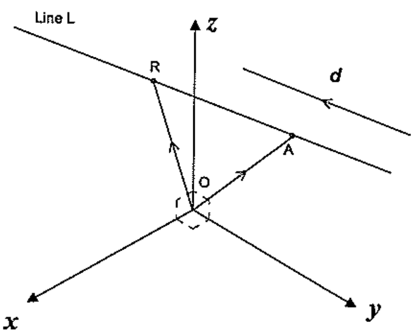
Solve I and II simultaneously: $\lambda = 1, \mu = -2$.



Substitute $\lambda = 1, \mu = -2$ into (III),
a true statement is obtained.
Hence, the two lines meet at $< 0, 3, 4 >$.

02 Vector Equation of Line & Plane

2.1 Vector Equation of a Line



- Consider the line L which passes through the fixed point A and which is parallel to vector $\boldsymbol{d}$.
- The point A has position vector **OA**.
- Let R be a variable point on the line L. The position vector of R is **OR**.
- Clearly **OR** = **OA** + **AR**.
- Since line L is parallel to $\boldsymbol{d}$, **AR** must be parallel to $\boldsymbol{d}$. Hence, **AR** $= \lambda\, \boldsymbol{d}$.
- Therefore, **OR** = **OA** $+ \lambda\, \boldsymbol{d}$.
  That is, the position vector of any point on the line L can be written in this form.
- Alternatively, the equation **OR** = **OA** $+ \lambda\, \boldsymbol{d}$ describes the position vector of any point on the line passing through the fixed point A and parallel to vector $\boldsymbol{d}$. As λ changes, the point R traces a line parallel to $\boldsymbol{d}$ passing through the point A.
- Rewrite **OR** $= \boldsymbol{r}$ and **OA** $= \boldsymbol{a}$. The vector equation of a line passing through the fixed point with position vector $\boldsymbol{a}$ and parallel to $\boldsymbol{d}$ is given by:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\, \boldsymbol{d}$$

position vector of any point on line ($\boldsymbol{r}$)

position vector of fixed point ($\boldsymbol{a}$)

direction vector of line ($\boldsymbol{d}$)

- Hence, the form of the equation is identical to that in the two-dimensional case.

Example 2.1

Find the vector equation of the line passing through the point with position vector $-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}$ and:

(a) parallel to the vector $\boldsymbol{i} - \boldsymbol{j} + 2\,\boldsymbol{k}$

(b) the point with position vector $\boldsymbol{i} + 6\,\boldsymbol{j} - \boldsymbol{k}$.

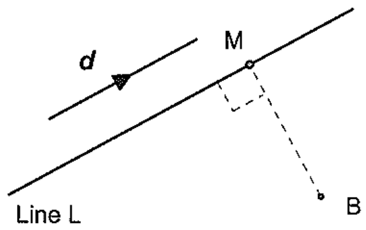
Solution:

(a) Vector equation of line is $\boldsymbol{r} = (-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + 6\,\boldsymbol{k}) + \lambda(\boldsymbol{i} - \boldsymbol{j} + 2\,\boldsymbol{k})$

$$= (-2 + \lambda)\,\boldsymbol{i} + (3 - \lambda)\,\boldsymbol{j} + (6 + 2\lambda)\,\boldsymbol{k}$$

2.1.3 Shortest Distance between point and line

- Let M be a point on line L with equation $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$.
- Let B with position vector $\boldsymbol{b}$, be a point not on line L.
- When B is closest to line L, M is the foot of the perpendicular from B to line L.
- Hence, when B is closest to line L, **BM** is perpendicular to line L. That is, **BM** is perpendicular to $\boldsymbol{d}$ (which is the direction vector of line L).
- Hence, the closest distance is $|\mathbf{BM}|$ where $\mathbf{BM} \bullet \boldsymbol{d} = 0$.



Example 2.7

Use a vector method to find the minimum distance between the point P with position vector $< 2, 1, 4 >$ and the line $\boldsymbol{r} = < 2 + \lambda, -\lambda, -1 - \lambda >$.

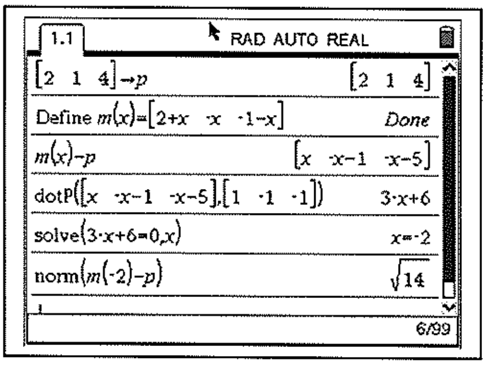
Solution:

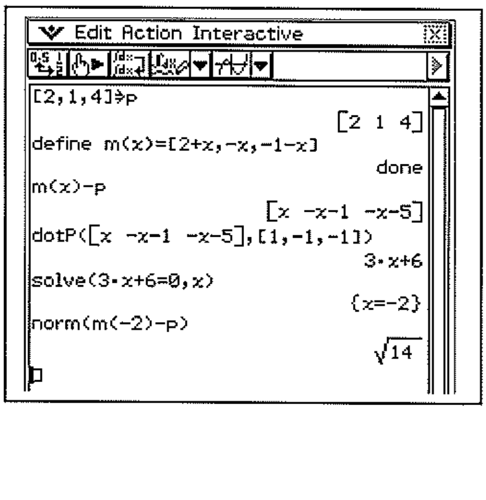
Let M with position vector $< a, b, c >$ be a point on the line $\boldsymbol{r} = < 2 + \lambda, -\lambda, -1 - \lambda >$.
Hence, $< a, b, c > = < 2 + \lambda, -\lambda, -1 - \lambda >$.
This gives: $a = 2 + \lambda$
$b = -\lambda$
$c = -1 - \lambda$

Also, $\mathbf{PM} = < a, b, c > - < 2, 1, 4 >$
$= < 2 + \lambda, -\lambda, -1 - \lambda > - < 2, 1, 4 >$
$= < \lambda, -\lambda - 1, -\lambda - 5 >$

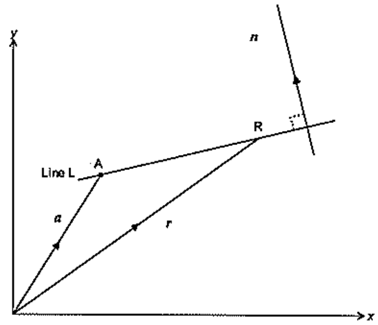
P is closest to the given line when **PM** is perpendicular to its direction vector $\boldsymbol{d}$.
Direction vector of line is $\boldsymbol{d} = < 1, -1, -1 >$.
$\Rightarrow \quad < \lambda, -\lambda - 1, -\lambda - 5 > . < 1, -1, -1 > = 0$
$\Rightarrow \lambda = -2$

Hence, shortest distance between P and given line
$= |< \lambda, -\lambda - 1, -\lambda - 5 >|$
$= |< -2, 1, -3 >|$
$= \sqrt{14}$





2.2 Review: Scalar Product Equation of a Line in 2D



- Consider Line L which passes through the point A with position vector $\boldsymbol{a}$. Let Line L be perpendicular to vector $\boldsymbol{n}$.
- Let R with position vector $\boldsymbol{r}$ be any point on Line L.
- Clearly, **AR** is also perpendicular to $\boldsymbol{n}$.
- Hence, $\mathbf{AR} \cdot \boldsymbol{n} = 0$.
  But $\mathbf{AR} = \boldsymbol{r} - \boldsymbol{a}$. $\Rightarrow$ $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$
  $$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$
- In summary, the scalar product (dot product) equation of a line passing through the point with position vector $\boldsymbol{a}$ and perpendicular (normal) to the vector $\boldsymbol{n}$ is:

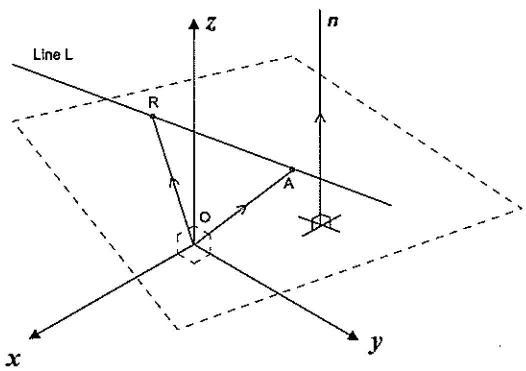
$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Position vector of any point on line

Position vector of fixed point on line

Vector perpendicular to line

2.3 Vector Equation of a Plane



- Consider the plane Π which passes through the fixed point A with position vector $\boldsymbol{a}$. Let vector $\boldsymbol{n}$ be perpendicular to the plane Π.
- Let point R with position vector $\boldsymbol{r}$ be any point on the plane Π. Note that the points A and R form a line L on the plane Π.
- Clearly, **AR** which is within the plane Π is also perpendicular to $\boldsymbol{n}$.
- Hence, $\mathbf{AR} \cdot \boldsymbol{n} = 0$.
  But $\mathbf{AR} = \boldsymbol{r} - \boldsymbol{a}$. $\Rightarrow$ $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$
  $$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$
- In summary, the vector equation of a plane passing through the point with position vector $\boldsymbol{a}$ and perpendicular (normal) to the vector $\boldsymbol{n}$ is:

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Position vector of any point on plane

Position vector of fixed point on plane

Vector perpendicular to plane

Note:

- The vector equation $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ is ambiguous and must be read in context.
- In a 2D context, it represents the vector equation (scalar product form) of a line passing through the fixed point with position vector $\boldsymbol{a}$ and perpendicular to vector $\boldsymbol{n}$.
- However, in a 3D context, it represents the vector equation of a plane passing through the fixed point with position vector $\boldsymbol{a}$ and perpendicular to vector $\boldsymbol{n}$.
- In each case, $\boldsymbol{n}$ is termed the *normal* vector.

Example 2.8

Given that the point $< 1, 1, 5 >$ lies on the plane $\boldsymbol{r} \cdot < 3, 1, -4 > = k$. Find k.

Solution:

Substitute $< 1, 1, 5 >$ into the equation of the plane, $< 1, 1, 5 > \cdot < 3, 1, -4 > = k$.

$$3 + 1 - 20 = k \Rightarrow k = -16.$$

Example 2.9

Determine if the points with position vectors $< 1, 1, 4>$ and $< 5, -1, 8 >$ lie on the plane with equation $\boldsymbol{r} \cdot < -1, 3, 2 > = 10$.

Solution:

Substitute $< 1, 1, 4>$ into LHS of equation of line. $\Rightarrow$ $< 1, 1, 4> \cdot < -1, 3, 2 > = 10$.
Hence, $< -1, 3, 2 >$ lies on the plane.

$< 5, -1, 8 > \cdot < -1, 3, 2 > = 8 \neq 10$. Hence, $< 5, -1, 8 >$ does not lie on the plane.

Example 2.10

Find the vector equation of the plane passing through the point with position vector $-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + \boldsymbol{k}$ and perpendicular to the vector $-3\,\boldsymbol{i} + 2\,\boldsymbol{k}$.

Solution:

Vector equation of plane is $\boldsymbol{r} \cdot (-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + \boldsymbol{k}) = (-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + \boldsymbol{k}) \cdot (-3\,\boldsymbol{i} + 2\,\boldsymbol{k})$

$$\Rightarrow \boldsymbol{r} \cdot (-2\,\boldsymbol{i} + 3\,\boldsymbol{j} + \boldsymbol{k}) = 8$$

Example 2.11

Find the vector equation of the plane perpendicular to the vector $< 2, -2, 5 >$ and containing the line with equation $\boldsymbol{r} = < 1 - 2\lambda, 3 + 4\lambda, -1 - \lambda >$.

Solution:

Let $\lambda = 0$.

Hence, the point with position vector $< 1, 3, -1 >$ lies on the given line and hence lies on the given plane.

Therefore, vector equation of plane is $\boldsymbol{r} \cdot < 1, 3, -1 > = < 1, 3, -1 > \cdot < 2, -2, 5 >$

$$\boldsymbol{r} \cdot < 1, 3, -1 > = -9$$

Example 2.12

Find the position vector of the point of intersection between the line $\boldsymbol{r} = < 2\lambda, 1 + \lambda, 3 - \lambda >$ and the plane $\boldsymbol{r} \cdot < -1, 1, 4 > = 3$.

Solution:

Substitute $\boldsymbol{r} = < 2\lambda, 1 + \lambda, 3 - \lambda >$ into $\boldsymbol{r} \cdot < -1, 1, 4 > = 5$.

$$< 2\lambda, 1 + \lambda, 3 - \lambda > \cdot < -1, 1, 4 > = 3$$

$$\Rightarrow \quad -5\lambda + 13 = 3 \quad \Rightarrow \quad \lambda = 2$$

Hence, point of intersection has position vector $< 4, 3, 1 >$.

Example 2.13

Find the vector equation of the plane passing through the points A, B and C with position vectors $< 1, 2, 1 >$, $< -2, -1, 4 >$ and $< 2, 1, -2 >$ respectively.

Solution:

$\mathbf{AB} = < -2, -1, 4 > - < 1, 2, 1 > = < -3, -3, 3 >$

$\mathbf{AC} = < 2, 1, -2 > - < 1, 2, 1 > = < 1, -1, -3 >$

Let $\boldsymbol{n} = < x, y, z >$ be the normal vector to the plane containing the points A, B and C.

Hence, $\mathbf{AB} \cdot < x, y, z > = 0 \quad \Rightarrow \quad < -3, -3, 3 > \cdot < x, y, z > = 0$

$$\Rightarrow \quad -3x - 3y + 3z = 0$$

$$-x - y + z = 0 \qquad \text{I}$$

Also, $\mathbf{AC} \cdot < x, y, z > = 0 \quad \Rightarrow \quad < 1, -1, -3 > \cdot < x, y, z > = 0$

$$\Rightarrow \quad x - y - 3z = 0 \qquad \text{II}$$

I + II gives: $\quad -2y - 2z = 0 \quad \Rightarrow \quad y = -z$

Substitute into I $\quad x = 2z$

Hence, any vector of the form $< 2t, -t, t >$ where t is real, will be normal to the plane.

Hence, vector equation of plane is $\boldsymbol{r} \cdot < 2, -1, 1 > = < 1, 2, 1 > \cdot < 2, -1, 1 >$

$$\boldsymbol{r} \cdot < 2, -1, 1 > = 1$$